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Some Properties of Subideals of Jordan Triple Systems

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Abstract: In this article, after giving some suitable concepts and elementary properties of subideals of a Jordan triple system, the authors show that any regular subideal of a Jordan triple system is a regular ideal, and further, any subideal of a regular Jordan triple system is a regular ideal.

Key words: jordan triple system; subideal; regular

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1 Preliminaries

Throughout the present paper, we assume ϕ is a commutative ring with 1.

Definition 1.1 A unital ϕ-moudule T together with a trilinear map $T \times T \times T \to T, (x, y, z) \mapsto \{xyz\}$ is called a **triple system**.

Definition 1.2 A map of unital ϕ-module $P: M \to M'$ is called **quadratic**, if

1) $P(ax) = a^2 P(x)$ for all $x \in M, a \in \phi$, and

2) $P(x, y) := P(x+y) - P(x) - P(y)$ is linear in x and y.

Let J be a unital ϕ-module and $P: J \to End\, J$ be a quadratic map. In this case, we call (J, P) **a quadratic triple system**, since P induces a trilinear composition $(x, y, z) \mapsto \{xyz\} := P(x, z)y$, and hence P induces on J the structure of a triple system. Usually, we omit the adjective "quadratic" and simply call the pair (J, P) a triple system (Most often we call J rather than (J, P) the triple system.). And further, the map P induces a bilinear map $L: J \times J \to EndJ, (x, y) \mapsto L(x, y)$, where $L(x, y)z = \{xyz\} = P(x, z)y$.

If Ω is a unital commutative associative algebra over ϕ (i.e., an extension of ϕ), we denote $J_\Omega := \Omega \otimes_\phi J$. ($J_\Omega$ is an Ω-module, $\omega'(\omega \otimes j) = \omega'\omega \otimes j$.)

Definition 1.3 The pair (J, P) is called a **Jordan triple system (Jts)**, if

(J.T.1) $L(x, y)P(x) = P(x)L(y, x)$ "homotopy formula"

(J.T.2) $L(P(x)y, y) = L(x, P(y)x)$

(J.T.3) $P(P(x)y) = P(x)P(y)p(x)$ "fundamental formula"

hold in J and all extension J_Ω.

Definition 1.4 If J is a Jordan triple system over ϕ, a submodule A of J is *Jt-subsystem*, if $P(A)A \subseteq A$; It is

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a **Jt-ideal** and denoted by $A \lhd J$ or $J \rhd A$, if $P(A)J \subseteq A$, $P(J)A \subseteq A$ and $\{JJA\} \subseteq A$. Let (J, P), (J', P') be triple systems over ϕ, a ϕ-linear map $f: J \longrightarrow J'$ is a **homomorphism** from J to J', if $f(P)(x)y) = P'(f(x))f(y)$ for all $x, y \in J$. **Ismorphism** and **automorphism** are defined in the usual way. As usual, A is an ideal, iff it is the kernel of some homomorphism, and the **quotient** $\bar{J} = J/A$ (A is an ideal of J) together with the induced map $\bar{P}$, $\bar{P}(\bar{x})\bar{y} = \overline{P(x)y}$, is a Jts. The usual homomorphism and ismorphism theorems hold.

In the following, we denote by J a Jordan triple system over ϕ.

Definition 1.5 J is called a **simple Jordan triple system**, if $P(J)J \neq 0$ and J has no proper ideals. An ideal of J is called **simple** if it itself is a simple triple system.

Definition 1.6 Let A be a subsystem of J, the **derived series** of A are defined recursively as

$$A^{\langle 1 \rangle} = A, \qquad A^{\langle 2(k+1)-1 \rangle} = P(A^{\langle 2k-1 \rangle})A^{\langle 2k-1 \rangle},$$

for $k = 1, 2, \cdots$

A is called **solvable**, if $A^{\langle 2k-1 \rangle} = 0$ for some $k \in N$. By the usual homomorphism and isomorphism theorems, we see that if A, B are solvable ideals of J, then $A + B$ is a solvable ideal, and if J is Noetherian, then J has a unique maximal solvable ideal **RadJ.** The unique maximal solvable ideal RadJ is called the **solvable radical** of J. J is called **semisimple**, if RadJ $= 0$.

Definition 1.7 Let A be a subsystem of J, The **powers** of A are defined recursively as

$$A^1 = A, \qquad A^{2(k+1)-1} = P(A^{2k-1})A + P(A)A^{2k-1} + \{AAA^{2k-1}\},$$

for $k = 1, 2, \cdots$

We call A **nilpotent**, if $A^{2k-1} = 0$ for some $k \in N$.

Definition 1.8 A subsystem A of J is called a **subideal** of J, if there exists $n+1$ subsystems A_i, for $i = 0, 1, 2, \cdots, n$, such that $A = A_0 \lhd A_1 \lhd A_2 \lhd \cdots \lhd A_{n-1} \lhd A_n = J$.

Remark: we denote " A is a subideal of J " by either " $A \lhd \lhd J$ or " $J \rhd \rhd A$ ".

2 Main results

For a Jordan triple system J, we have

Theoram 2.1 If $A \lhd \lhd J$, $B \lhd \lhd J$ and $D \lhd \lhd A$, J^* is an arbitrary subsystem of J, then

1) $D \lhd \lhd J$.

2) $(A \cap J^*) \lhd \lhd J^*$.

3) If $A \subseteq J^*$, then $\lhd \lhd J^*$.

4) $(A \cap B) \lhd \lhd J$.

5) If f is an epimorphism of J onto J', then $f(A) = A'$ is a subideal of J'; conversely, if E' is a subideal of J', and E is the complete inverse image of E' under f, then $E \lhd \lhd J$.

6) If J is nilpotent, then $J^* \lhd \lhd J$.

7) If J is the direct sum of its simple ideals, then all subideals of J are its ideals.

Proof. It is trivial to prove this theorem, so we omit it.

By definitsion 1.8, each ideal of a Jodan triple system is a subideal of it, but it is not necessarily true vice versa. As an example, we consider a special jordan triple system.

In [5], E. I. Zel′manov defined the structure of a Jordan triple system $R^{(+)}$ on an associative algebra R by setting $P(x)y = xyx$, and called it one of the eight sorts of classical Jordan triple systems.

Let M be a set of strongly upper triangular matrices $(a_{ij})_{5 \times 5}$ ($a_{ij} = 0$, *if* $i \geqslant j$) over real number field R. It is readily seen that M is an associative R-algebra with the customary matrix addition, multiplication and scalar operation by elments of R. We define $P(X)Y=XYX$ (multiplication of matrices), for all $X, Y\in M$. Then (M, P) is a Jts, denoted by $M^{(+)}$.

We notice that in $M^{(+)}$, $\{XYZ\}=L(X,Y)Z=P(X,Z)Y=P(X+Z)Y-P(X)Y-P(Z)Y=XYZ+ZYX$, $\forall X,Y,Z\in M^{(+)}$.

It is easy to prove that M is a nilpotent associative algebra. In the same way, we can show that $M^{(+)}$ is a nilpotent Jts and

$$M_1=\left\{\begin{bmatrix}0 & a_{12} & a_{13} & a_{14} & 0\\ 0 & 0 & a_{23} & a_{24} & 0\\ 0 & 0 & 0 & a_{34} & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}, a_{ij}\in R\right\},$$

is a Jt-subsystem of $M^{(+)}$.

By theorem 2.1.6), we have $M_1\lhd M_1+M^3\lhd M$.

Let

$$m_1=\begin{bmatrix}0&1&1&1&0\\0&0&1&1&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\in M_1,\ m=\begin{bmatrix}0&1&1&1&1\\0&0&1&1&1\\0&0&0&1&1\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}\in M,$$

then $\{m_1mm\}\in\{M_1MM\}$.

But

$$\{m_1mm\}=\begin{bmatrix}0&0&0&2&3\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\notin M_1.$$

Thus, M_1 is a subideal of M, but M_1 is not an ideal of M.

By the above-mentioned example, someone may ask: When a subideal of a Jts is its dieal. In Theorem2.6), 7), we have two results for this questing. In the following, we will give some more conclusions for it.

At first, we introduce a kind of important Jordan triple system, i.e., regular Jordan triple system.

Definition 2.2[1] $x\in J$ is called **regular**, if $x=P(x)u$ for some $u\in J$; J is called a **regular Jts**, if each element in it is regular; a subsystem A of J is called **regular**, if A is regular as a Jts; an ideal B of J is called **regular**, if each element in it is regular.

Lemma 2.3[1] If $x\in J$ is regular, then there exists $y\in J$ such that (x, y) is a regular pair, i.e., $P(x)y=x$, $P(y)x=y$.

Lemma 2.4 If J is a Jts over ϕ, B is an ideal of J, then the following two properties are equivalent:

1) B is a regular Jt-subsystem of J;

2) Each element in B is regular in J.

Proof. 1)⇒2) is trivial.

2)⇒1): Let $x\in B$. since x is regular in J, hence by lemma2.3, there exists $y\in J$ such that $x=P(x)y$, and $y=P(y)x$. Thus we get $y\in B$, since B is an ideal of J. Therefore x is regular in B, i.e., B is a regular subsystem of J. This completes the Proof of the lemma.

Remark: By lemma2.4. If $B(\subseteq J)$ is an ideal of J, then B is regular iff B is a regular subsystem of J.

Theorem 2.5 if J is a Jts over ϕ, and A is a regular subideal of J (i.e., A is both a regular subsystem and a subideal of J), then A is a regular ideal of J.

Proof. Since $A \triangleleft \triangleleft J$, we have the following chain of subideals: $A = A_0 \triangleleft A_1 \triangleleft A_2 \triangleleft \cdots \triangleleft A_r = J$.

First, we show that $A \triangleleft A_2$. To do so, it suffices to show 1). $P(A)A_2 \subseteq A$; 2). $\{A_1 A_2 A\} \subseteq A$; 3). $P(A_2)A \subseteq A$.

Let $a \in A$, $x, y \in A_2$. Since A is regular, we have $b, c \in A$ such that $a = P(a)b$ and $b = P(b)c$, thus $a = P(a)P(b)c$. Then $P(a)A_2 = P(a)P(b)P(a)A_2 \subseteq P(a)P(b)A_1 \subseteq A$, since A_2 is an ideal of A_1 and A_1 is an ideal of A. Hence 1) is true.

To prove 3), we have $xya = L(x, y)P(a)b = P(\{xyz\}, a)b - P(a)\{yxb\} \in P(A_1, A)A + P(A, A_1) \subseteq A$, for all $a \in A$, $x, y \in A_2$.

Finally, 2) is true since

$P(x)a = P(x)P(a)P(b)c = P(\{xab\})c + P(P(x)a, P(b)a)c - P(b)P(a)P(x)c - P(x, b)P(a)P(x, b)c \in A$, for all $a \in A$, $x, y \in A_2$.

Hence, we get $A \triangleleft A_2$.

Thus, the above-mentioned chain of subsystems becomes $A = A_2 \triangleleft A_3 \triangleleft \cdots \triangleleft A_r = J$. In the same way, we have $A \triangleleft A_3$, then we can get $A \triangleleft J$ by induction. By Lemma2.4, we see that A is a regular ideal of J, since A is a regular subsystem. This completes the proof of the theorem.

Corollary 2.6 If (J, P) is a regular Jts over ϕ, and A is a subideal of J, then A is a regular ideal of J.

Proof. By Theorem2.5, it suffices to show that A a regular subsystem of J. In fact, since $A \triangleleft \triangleleft J$, we get $A = A_0 \triangleleft A_1 \triangleleft A_2 \triangleleft \cdots \triangleleft A_r = J$. Since $A_{r-1} \triangleleft J$, and J is a regular Jts, by definition2.2, A_{r-1} is a regular ideal of J. Thus by Iemma2.4, A_{r-1} is a regular subsystem of J. In the same way, we have that A_{r-2} is a regular subsystem of A_{r-1}. Then by induction we get that A is a regular subsystem of A_1, i.e., A is regulsr as a *Jts*. Therefore, A is a regular subsystem of J. The corollary is proved.

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Jordan 三系的次理想的几个性质

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摘　要: 首先给出了 Jordan 三元系的次理想的定义与 Jordan 三元系的次理想的几条简单性质, 举例说明了次理想与理想是 2 个不同的概念. 然后, 对一类重要的 Jordan 三元系——正则 Jordan 三元系, 证明了它的次理想成为理想的几个充分条件.

关键词: Jordan 三元系; 次理想; 正则的

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